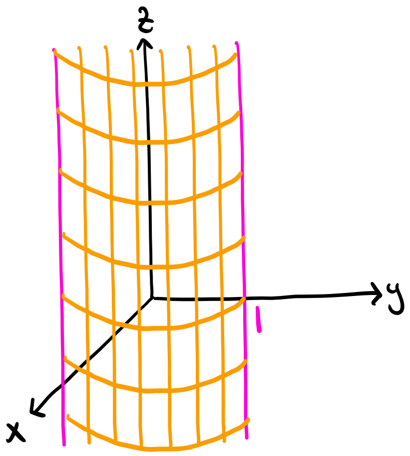
# 16.6. Parametric surfaces

**Def** (1) A parametric surface is an object parametrized by a vector function of two variables.

(2) A grid curve of a vector function $\vec{r}(u,v)$ is given by setting either $u$ or $v$ to be constant.

e.g. The cylinder $x^2+y^2=1$ is parametrized by




$$\vec{r}(\theta, z) = (\cos\theta, \sin\theta, z)$$

$\theta$ constant $\Rightarrow$ vertical lines

$z$ constant $\Rightarrow$ circles.

★ **Note** The graph $z=f(x,y)$ is parametrized by

$\vec{r}(x,y) = (x, y, f(x,y))$ "xy-parametrization".

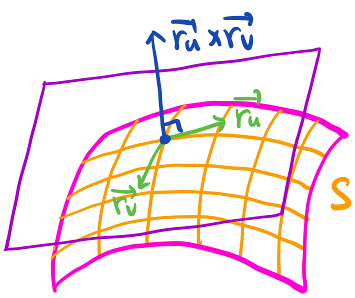
**Prop** Consider a vector function

$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$,

(1) The partial derivatives of $\vec{r}(u,v)$ are

$$\vec{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \text{ and } \vec{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

★ (2) If a surface $S$ is parametrized by $\vec{r}(u,v)$, then the tangent plane to $S$ has a normal vector $\vec{r}_u \times \vec{r}_v$.




$\vec{r}_u$ and $\vec{r}_v$ are tangent vectors of the grid curves

**Ex** Sketch the surface parametrized by

$$\vec{r}(u,v) = (2u\cos v,\ 2u\sin v,\ v) \quad \text{with } 1 \le u \le 3,\ 0 \le v \le \pi.$$

**Sol** Idea : Sketch grid curves.

$u=1 \Rightarrow \vec{r}(1,v) = (2\cos v,\ 2\sin v,\ v)$

$\rightsquigarrow$ a helix from $(-2,0,0)$ to $(2,0,\pi)$

$u=2 \Rightarrow \vec{r}(2,v) = (4\cos v,\ 4\sin v,\ v)$

$\rightsquigarrow$ a helix from $(-4,0,0)$ to $(4,0,\pi)$

$u=3 \Rightarrow \vec{r}(3,v) = (6\cos v,\ 6\sin v,\ v)$

$\rightsquigarrow$ a helix from $(-6,0,0)$ to $(6,0,\pi)$

$v=0 \Rightarrow \vec{r}(u,0) = (-u,0,0)$

$\rightsquigarrow$ a line segment from $(-1,0,0)$ to $(-3,0,0)$

$v=\frac{\pi}{3} \Rightarrow \vec{r}(u,\frac{\pi}{3}) = (\frac{u}{2},\ \frac{\sqrt{3}}{2}u,\ \frac{\pi}{3})$

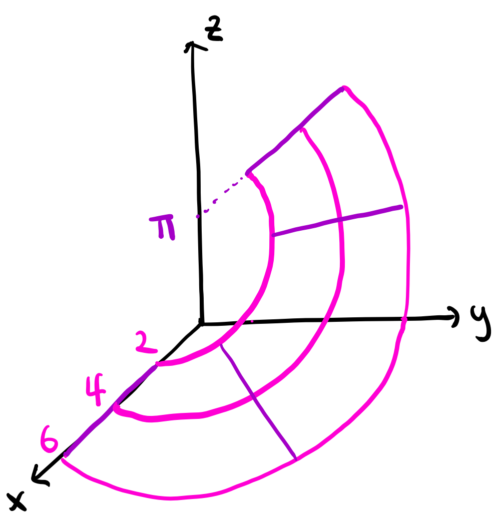
$\rightsquigarrow$ a line segment from $(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3})$ to $(\frac{3}{2}, \frac{3\sqrt{3}}{2}, \frac{\pi}{3})$

$v=\frac{2\pi}{3} \Rightarrow \vec{r}(u,\frac{2\pi}{3}) = (-\frac{u}{2},\ \frac{\sqrt{3}}{2}u,\ \frac{2\pi}{3})$

$\rightsquigarrow$ a line segment from $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{5\pi}{3})$ to $(\frac{3}{2}, -\frac{3\sqrt{3}}{2}, \frac{5\pi}{3})$

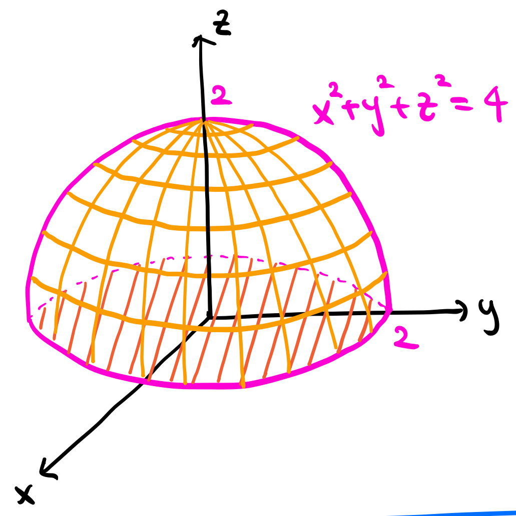
$v=\pi \Rightarrow \vec{r}(u,\pi) = (-u,0,\pi)$

$\rightsquigarrow$ a line segment from $(-1,0,\pi)$ to $(-3,0,\pi)$.



**Ex** Find a parametrization of the hemisphere $x^2+y^2+z^2=4$ with $z \geq 0$.

**Sol 1**




$x^2+y^2+z^2=4 \rightsquigarrow z=\sqrt{4-x^2-y^2}$ $(z \geq 0)$

The shadow on the xy-plane is given by $x^2+y^2 \leq 4$.

$$\Rightarrow \boxed{\vec{r}(x,y) = (x, y, \sqrt{4-x^2-y^2}) \text{ with } x^2+y^2 \leq 4.}$$

**Sol 2** In cylindrical coordinates:

$x^2+y^2+z^2=4 \rightsquigarrow r^2+z^2=4 \rightsquigarrow z=\sqrt{4-r^2}$ $(z \geq 0)$

$\Rightarrow x = r\cos\theta,\ y = r\sin\theta,\ z=\sqrt{4-r^2}$

The shadow on the xy-plane: $0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2$

$$\Rightarrow \boxed{\vec{s}(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{4-r^2}) \text{ with } 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2}$$

**Sol 3** In spherical coordinates:

$x^2+y^2+z^2=4 \rightsquigarrow \rho^2=4 \rightsquigarrow \rho=2.$

$\Rightarrow x = 2\sin\varphi\cos\theta,\ y = 2\sin\varphi\sin\theta,\ z=2\cos\varphi.$

$z \geq 0 \Rightarrow 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \frac{\pi}{2}$

$$\Rightarrow \boxed{\begin{gathered}\vec{t}(\theta,\varphi) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi) \\ \text{with } 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \frac{\pi}{2}\end{gathered}}$$

**Ex** Find an equation of the tangent plane to the paraboloid

$z = x^2 + y^2$ at $(1, 1, 2)$.

**Sol 1** (Using the gradient)

$z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0$

$\rightsquigarrow$ a level surface of $f(x, y, z) = x^2 + y^2 - z$.

$\nabla f = (f_x, f_y, f_z) = (2x, 2y, -1)$

A normal vector is $\nabla f(1, 1, 2) = (2, 2, -1)$

The tangent plane at $(1, 1, 2)$ is given by

$$\boxed{2(x-1) + 2(y-1) - (z-2) = 0}$$

**Sol 2** (Using a parametrization)

The paraboloid $z = x^2 + y^2$ is parametrized by

$\vec{r}(x, y) = (x, y, x^2 + y^2)$

$\Rightarrow \vec{r_x} = (1, 0, 2x), \quad \vec{r_y} = (0, 1, 2y)$

$\Rightarrow \vec{r_x} \times \vec{r_y} = (-2x, -2y, 1)$

At $(1, 1, 2)$ : $\vec{r_x} \times \vec{r_y} = (-2, -2, 1)$

The tangent plane at $(1, 1, 2)$ is given by

$$\boxed{-2(x-1) - 2(y-1) + (z-2) = 0}$$

**Note** You can also use a cylindrical parametrization

$\vec{S}(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$ with $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$ at $(1, 1, 2)$.

However, the computation is quite tedious.

Ex Let $S$ be the surface parametrized by

$\vec{r}(u,v) = (u^3+1, v^2+1, u+v)$ with $u, v > 0$.

(1) Find an equation of the tangent plane to $S$ at $(2,5,3)$

Sol $\vec{r_u} = (3u^2, 0, 1)$ and $\vec{r_v} = (0, 2v, 1)$

$\Rightarrow \vec{r_u} \times \vec{r_v} = (-2v, -3u^2, 6u^2v)$

Find $u$ and $v$ at $(2,5,3)$.

$\vec{r}(u,v) = (2,5,3)$

$\Rightarrow u^3+1 = 2,\ v^2+1 = 5,\ u+v = 3 \Rightarrow u=1,\ v=2$

* $v=-2$ works for the second equation, but not for the last equation.

A normal vector is $\vec{r_u} \times \vec{r_v} = (-4, -3, 12)$.

The tangent plane at $(2,5,3)$ is given by

$$\boxed{-4(x-2)-3(y-5)+12(z-3) = 0}$$

(2) Find all points on $S$ where the tangent plane is parallel to the xy-plane.

Sol If the tangent plane is parallel to the xy-plane, the normal vector $\vec{r_u} \times \vec{r_v} = (-2v, -3u^2, 6u^2v)$ must be parallel to $\vec{k} = (0,0,1)$

$\Rightarrow -2v = 0,\ -3u^2 = 0 \Rightarrow \boxed{\text{no solutions}}$

$u, v > 0$